GAUSSIAN PROCESSES
EXERCISE SHEET 11: INFINITE DIMENSIONAL MEASURES

Exercise 1.

(a). Since $\mathrm{Leb}([a,b]) = b-a$, we have

$$\mathrm{Leb}_{\otimes\mathbb{N}}([a,b]^{\mathbb{N}}) = \prod_{n=1}^{\infty}(b-a).$$

Thus

$$\mathrm{Leb}_{\otimes\mathbb{N}}([a,b]^{\mathbb{N}}) = \begin{cases} 0, & b-a<1, \\ 1, & b-a=1, \\ +\infty, & b-a>1. \end{cases}$$

(b). The metric is

$$d(x,0) = \sum_{n=1}^{\infty} 2^{-n}\frac{|x_n|}{1+|x_n|}.$$

Fix $r>0$ and choose $m \in \mathbb{N}$ such that $4 \cdot 2^{-m} < r$. If $|x_n| \le 2^{-m}$ for each $n \le m$, then

$$d(x,0) \le \sum_{n=1}^{m} 2^{-n}\frac{2^{-m}}{1+2^{-m}} + \sum_{n=m+1}^{\infty} 2^{-n} < 2^{-m+1} + 2^{-m} < r.$$

Hence the cylinder set

$$[-2^{-m},\, 2^{-m}]^{\{1,\dots,m\}} \;\times\; \mathbb{R}^{\{m+1,m+2,\dots\}}$$

is contained in $B(0,r)$. Its product measure is

$$\prod_{n=1}^{m}\mathrm{Leb}([-2^{-m},2^{-m}]) \cdot \prod_{n>m}\mathrm{Leb}(\mathbb{R}) = (2^{1-m})^m \cdot \infty = \infty.$$

Thus

$$\mathrm{Leb}_{\otimes\mathbb{N}}(B(0,r)) = \infty, \qquad r>0.$$

(c). One possible additional assumption is local finiteness, which contradicts (b). Another natural choice is outer regularity. By outer regularity, for any set E of finite measure (take E to be $[0,1]^\infty$ for example),

$$\mathrm{Leb}_{\otimes\mathbb{N}}(E) = \inf\{\mathrm{Leb}_{\otimes\mathbb{N}}(U) : E \subset U,\ U \text{ open}\}.$$

Hence we can always find an open set U containing E with finite measure, and, in particular, an open ball contained in U that also has finite measure. This again contradicts (b).

□

Exercise 2.

Let $\Gamma(v)$ be i.i.d. Gaussian, and define $U_a = \{v : \Gamma(v) > a\}$. Consider

$$A = \{\text{there exists at least one infinite connected component in the induced subgraph}\}.$$

Measurability. Whether a given vertex v is connected to the boundary of a large box depends only on finitely many coordinates, so the event that v has an infinite cluster is a countable combination of cylinder events and hence measurable. Then

$$A = \bigcup_{v \in \mathbb{Z}^d} \{v \text{ has an infinite cluster}\}$$

is measurable.

0–1 Law. Changing $\Gamma(v)$ at finitely many vertices changes at most finitely many edges and therefore cannot create or destroy an infinite cluster. Thus A is tail measurable:

$$A \in \bigcap_{\substack{F \subset \mathbb{Z}^d \\ F \text{ finite}}} \sigma(\Gamma(v) : v \notin F).$$

By Kolmogorov's 0–1 law,

$$\mathbb{P}(A) \in \{0, 1\}.$$

□

Exercise 3.

Let $\{X_t\}_{t \in [0,1]}$ satisfy the finite-dimensional distributions of Brownian motion but not necessarily continuity. For each n, let

$$D_n = \{k2^{-n} : k = 0, 1, \dots, 2^n\}.$$

Construction using dyadic endpoints only.

Define a dyadic step-function approximation by

$$\tilde{X}_t^{(n)} := X_{\lfloor 2^n t \rfloor 2^{-n}}, \qquad t \in [0, 1].$$

Define

$$\tilde{X}_t := \lim_{n \to \infty} \tilde{X}_t^{(n)},$$

which should exist almost surely on $[0, 1]$ because Brownian increments vanish on small intervals.

Measurability. Each $\tilde{X}^{(n)}$ depends only on the finite set $\{X_{k2^{-n}}\}_{k=0}^{2^n}$, hence the map

$$X_\cdot \mapsto \tilde{X}_\cdot^{(n)}$$

is measurable. Since $\tilde{X}$ is the pointwise limit of measurable maps, the map $X_\cdot \mapsto \tilde{X}_\cdot$ is measurable.

General remark. Let $\Omega_0 \subset \Omega$ be the event of probability 1 on which the continuous modification $\tilde{X}_t$ is well-defined. We can extend $\tilde{X}_t$ to the whole space by setting it to 0 on the null set $\Omega \setminus \Omega_0$:

$$\tilde{X}_t(\omega) := \mathbf{1}_{\Omega_0}(\omega)\, \tilde{X}_t(\omega) + \mathbf{1}_{\Omega \setminus \Omega_0}(\omega) \cdot 0 = \begin{cases} \tilde{X}_t(\omega), & \omega \in \Omega_0, \\ 0, & \omega \notin \Omega_0. \end{cases}$$

Since Ω_0 is measurable and $\tilde{X}_t$ is measurable on Ω_0, this extension is measurable on the whole probability space Ω. The choice of 0 on the null set does not affect the law of the process or any almost-sure property.

For more details on an example, one may consult Kolmogorov's continuity theorem, which provides a systematic way to obtain continuous modifications.

□